Risk Flow Patterns

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Motivation

Cash flow patterns...

- ... help to determine "what part of our reserves becomes payable between k and ℓ years from now?"
  - liquidity mgmt, ALM, duration matching, discounting, IFRS 4 & 17
- ... are considered as characteristics of lines of business
  - benchmarking, regulatory use (e.g. FINMA SST patterns)
- ... have nice properties:
  - volume-independent, transform naturally upon change in time granularity.

Can we have something similar for the risk ???

Quick summary / Preview of Main Result

In a chain ladder model based on paid losses, looking at the development between k and ℓ accounting years from now, we may use the following predictors/estimators for...

... the cash flow:

$$\text{cash flow} \quad \approx \quad \hat{C} \sum_{j=1}^{J} \hat{\pi}_j \left(\hat{q}_{j-k} - \hat{q}_{j-\ell}\right)$$

... the squared prediction error of the loss development result:

$$\text{MSEP} \quad \approx \quad \hat{C} \sum_{j=1}^{J} \hat{\rho}_j \left(\frac{1}{1-\hat{q}_{j-k}} - \frac{1}{1-\hat{q}_{j-\ell}}\right)$$

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The Chain Ladder Method

Basic Notation

$C_{i,j} > 0$ is the cumulative paid loss from accident year i at development step j, where $i, j \in \{0, \dots, J\}$.

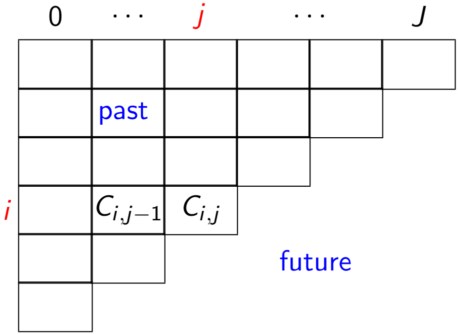
The values $C_{i,j}$ known today form a loss development triangle.

Ultimates at $j = J$.

Link ratios $f_{i,j} := C_{i,j}/C_{i,j-1}$.

Chain Ladder Principle: predict future values by

$$\hat{C}_{i,j} := \begin{cases} C_{i,j} & \text{if known,} \\ \hat{f}_j \hat{C}_{i,j-1} & \text{else.} \end{cases}$$

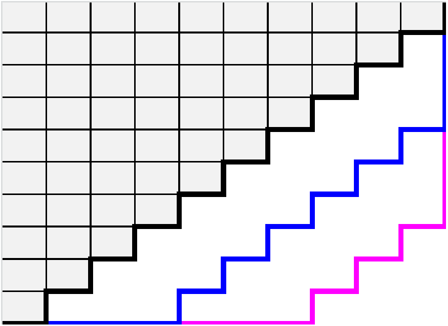


Development Factor Estimator

Use $\hat{f}_j := C_{\mathcal{I}_j,j}/C_{\mathcal{I}_j,j-1}$ where

$\mathcal{I}_j := \{i | i + j \leq J\}$,
$C_{\mathcal{H},j} := \sum_{i \in \mathcal{H}} C_{i,j}$.

Chain Ladder Predictors



Data today $\mathcal{D}$

Data k yrs from today $\mathcal{D}^{[k]}$

Data ℓ yrs from today $\mathcal{D}^{[\ell]}$

...“Horizons”

- From $\mathcal{D}$, get CL predictor $\hat{C} := \hat{C}_{\mathcal{I}_0,J}$ for ultimate loss $C := C_{\mathcal{I}_0,J}$
- From $\mathcal{D}^{[k]}$, will get predictor $\hat{C}^{[k]}$; from $\mathcal{D}^{[\ell]}$, predictor $\hat{C}^{[\ell]}$
- Can you suggest a predictor for the random variable $\hat{C}^{[\ell]} - \hat{C}^{[k]}$?

Prediction Error

Predict future development result $\hat{C}^{[\ell]} - \hat{C}^{[k]}$ by 0!

What is the prediction error ?

Definition (conditional) Mean Squared Error of Prediction (MSEP)

Predicting random variable X — given $\mathcal{D}$ — by predictor $\hat{X}$,

$$
\begin{aligned}
\mathrm{MSEP}_{X,\hat{X}} &:= E[(X-\hat{X})^2|\mathcal{D}] \\
&= E[(X-E[X|\mathcal{D}])^2|\mathcal{D}] + (E[X|\mathcal{D}]-\hat{X})^2 \\
&= V[X|\mathcal{D}] \quad + \quad (E[X|\mathcal{D}]-\hat{X})^2 \\
&= (\text{process error})^2 \quad + \quad (\text{parameter error})^2
\end{aligned}
$$

This (standard) definition only makes sense after specifying an underlying stochastic model. We use Mack's (1993) model.

Chain Ladder Processes

Mack's Stochastic Model (1993)

A chain ladder process is a discrete-time, real-valued stochastic process $\{X_j > 0\}_{j \geq 0}$, such that for each $j > 0$

$$\begin{aligned} E\left[X_j | X_{j-1}, \ldots, X_0\right] &= f_j \, X_{j-1}, \\ V\left[X_j | X_{j-1}, \ldots, X_0\right] &= \phi_j \, X_{j-1} \end{aligned}$$

with parameters $f_j > 0$ (development factors) and $\phi_j \geq 0$.

- Standard estimators from loss triangle $(1 \leq j \leq J)$:

$$\hat{f}_j := \frac{C_{\mathcal{I}_j, j}}{C_{\mathcal{I}_j, j-1}}, \qquad \hat{\phi}_j := \frac{\sum_{i \in \mathcal{I}_j} C_{i,j-1} \left(f_{i,j} - \hat{f}_j \right)^2}{-1 + \sum_{i \in \mathcal{I}_j} 1}$$

- Note that

$$V\left[X_j | X_{j-1}, \ldots, X_0\right] = \frac{\phi_j}{f_j} E\left[X_j | X_{j-1}, \ldots, X_0\right]$$

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Cash Flow Patterns and Risk Flow Patterns

Proposition

Assume the chain ladder process $\{X_j\}_{j \geq 0}$ becomes constant after step J (i.e. $f_j = 1$ and $\phi_j = 0$ for $j > J$). Then

$$E[X_J - X_{j-1} | X_{j-1}, \ldots, X_0] = (\pi_j + \pi_{j+1} + \ldots + \pi_J) E[X_J | X_{j-1}, \ldots, X_0]$$
$$V[X_J | X_{j-1}, \ldots, X_0] = (\rho_j + \rho_{j+1} + \ldots + \rho_J) E[X_J | X_{j-1}, \ldots, X_0]$$

where $\Pi_j := f_{j+1} \cdot \ldots \cdot f_J$, $\pi_j := \Pi_j^{-1} - \Pi_{j-1}^{-1}$ and $\rho_j := \Pi_j \phi_j / f_j$.

- It pays to express everything in terms of the expected ultimate.
- $\pi_j =:$ cash flow pattern.
- We call the ρ_j the risk flow pattern.
- The ρ_j have the same dimension as the X_j.
- Get estimators $\hat{\pi}_j$, $\hat{\rho}_j$ via $\hat{f}_j$, $\hat{\phi}_j$.
- Both patterns behave nicely upon change of time granularity.

Influence factors

Pattern values $\hat{\pi}_j$ will be multiplied by ultimates $\hat{C}_{i,J}$.

Instead of dealing with the $\hat{C}_{i,J}$ indexed by accident year i, it is convenient to work with percentages $\hat{q}_j$ of the total predicted ultimate loss $\hat{C}$:

$$\begin{aligned}
\hat{q}_j &:= \frac{\text{Predicted ultimate loss for the } j \text{ most recent accident years}}{\hat{C}} \\
&= \frac{\sum_{i=J-j+1}^{J} \hat{C}_{i,J}}{\hat{C}} \\
&= \text{percentage of } \hat{C} \text{ influenced by } \hat{f}_j \\
&= \frac{\partial \log[\hat{C}]}{\partial \log[\hat{f}_j]}
\end{aligned}$$

We call these $\hat{q}_j$ the "influence factors".

Example (Mack)

4370	6293	10292	12460	13660	14307
2701	5291	7162	8945	9338	9780
4483	6729	10074	11142	11971	12538
3254	5804	8351	9874	10608	11111
8010	12118	18028	21315	22901	23986
5582	8864	13187	15592	16752	17546

$\hat{f}_j =$	1.588	1.488	1.182	1.074	1.047
$\hat{q}_j =$	20%	47%	59%	73%	84%
$\hat{\pi}_j =$	18.7%	24.6%	13.7%	6.6%	4.5%
$\hat{\rho}_j =$	209.1	73.6	47.0	13.9	3.9

From $\mathcal{D}$, get...

- link ratios $f_{i,j}$;
- estimator $\hat{f}_j$ for f_j;
- predicted loss development $\hat{C}_{i,j}$;
- influence factors $\hat{q}_j$
- cash flow pattern $\hat{\pi}_j$ (N.B.: $\hat{\pi}_0 = 31.8\%$ not shown here)
- risk flow pattern $\hat{\rho}_j$

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Main Result

Looking at the development between k and ℓ accounting years from now, $0 \leq k \leq \ell$, we may use the following predictors/estimators for...

... the cash flow:

$$\text{cash flow} \approx \hat{C} \sum_{j=1}^{J} \hat{\pi}_j \left(\hat{q}_{j-k} - \hat{q}_{j-\ell} \right)$$

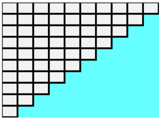
Proof: immediate from the definitions.

... the squared prediction error of the loss development result:

$$\text{MSEP}_{\hat{C}^{[k]} - \hat{C}^{[\ell]},0} \approx \hat{C} \sum_{j=1}^{J} \hat{\rho}_j \left(\frac{1}{1 - \hat{q}_{j-k}} - \frac{1}{1 - \hat{q}_{j-\ell}} \right)$$

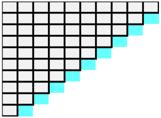
Proof: see Röhr (2016).

MSEP Formulae Based on Mack's Model



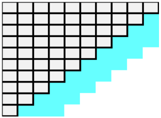
Mack 1993

$k = 0$ (today) $\longrightarrow$ $\ell = J$ (ultimate)



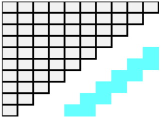
Merz/Wüthrich 2008

$k = 0$ (today) $\longrightarrow$ $\ell = 1$ (1 period from now)



Diers et al. 2016

$k = 0$ (today) $\longrightarrow$ ℓ periods from now



Our version (also Merz/Wüthrich 2014, Gisler 2016)

k periods from now $\longrightarrow$ ℓ periods from now

Mack (1993)

$$\widehat{mse}(\hat{R}_i) = \hat{C}_{iI}^2 \sum_{k=I+1-i}^{I-1} \frac{\hat{\sigma}_k^2}{\hat{f}_k^2} \left(\frac{1}{\hat{C}_{ik}} + \frac{1}{\sum_{j=1}^{I-k} C_{jk}} \right)$$

$$\widehat{mse}(\hat{R}) = \sum_{i=2}^{I} \left\{ (\text{s.e.}(\hat{R}_i))^2 + \hat{C}_{iI} \left(\sum_{j=i+1}^{I} \hat{C}_{jI} \right) \sum_{k=I+1-i}^{I-1} \frac{2\hat{\sigma}_k^2 / \hat{f}_k^2}{\sum_{n=1}^{I-k} C_{nk}} \right\}$$

Our version (algebraically identical) $k = 0, \ell = J$

$$\text{MSEP}_{C,\hat{C}} \approx \hat{C} \sum_{j=1}^{J} \hat{\rho}_j \left(\frac{1}{1-\hat{q}_j} - 1 \right)$$

Comparison with Merz/Wüthrich's Formula

Merz/Wüthrich (2008), see Bühlmann et al. (2009)

$$\widehat{\mathrm{msep}}_{\widehat{\mathrm{CDR}}_i(I+1)\,|\,\mathcal{D}_I}(0) \tag{4.19}$$

$$= \left(\widehat{C}_{i,J}^{CL}\right)^2 \left[\frac{\sigma_{I-i}^2 / \left(\widehat{F}_{I-i}^{(I)}\right)^2}{C_{i,I-i}} + \frac{\sigma_{I-i}^2 / \left(\widehat{F}_{I-i}^{(I)}\right)^2}{S_{I-i}^{[i-1]}} + \sum_{j=I-i+1}^{J-1} \frac{C_{I-j,j}}{S_j^{[I-j]}} \frac{\sigma_j^2 / \left(\widehat{F}_j^{(I)}\right)^2}{S_j^{[I-j-1]}} \right]$$

$$\widehat{\mathrm{msep}}_{\sum_{i=I-J+1}^{I} \widehat{\mathrm{CDR}}_i(I+1)\,|\,\mathcal{D}_I}(0) = \sum_{i=I-J+1}^{I} \widehat{\mathrm{msep}}_{\widehat{\mathrm{CDR}}_i(I+1)\,|\,\mathcal{D}_I}(0) \tag{4.20}$$

$$+ 2 \sum_{I-J+1 \le i < k \le I} \widehat{C}_{i,J}^{CL}\, \widehat{C}_{k,J}^{CL} \left[\frac{\sigma_{I-i}^2 / \left(\widehat{F}_{I-i}^{(I)}\right)^2}{S_{I-i}^{[i-1]}} + \sum_{j=I-i+1}^{J-1} \frac{C_{I-j,j}}{S_j^{[I-j]}} \frac{\sigma_j^2 / \left(\widehat{F}_j^{(I)}\right)^2}{S_j^{[I-j-1]}} \right].$$

Our version (algebraically identical) $k = 0, \ell = 1$

$$\mathrm{MSEP}_{\hat{C}^{[1]} - \hat{C},0} \approx \hat{C} \sum_{j=1}^{J} \hat{\rho}_j \left(\frac{1}{1 - \hat{q}_j} - \frac{1}{1 - \hat{q}_{j-1}} \right)$$

Comparison with Merz/Wüthrich's "Full Picture" Formula

Merz/Wüthrich (2014)

$$
\begin{aligned}
\varrho^{(I)}_{i,I+k+1} &= \widehat{\mathbb{E}}\left[\mathrm{msep}^{\mathrm{MW}}_{\mathrm{CDR}_{i,I+k+1}|\mathcal{D}_{I+k}}(0) \,\middle|\, \mathcal{D}_I \right] \\
&= \left(\widehat{C}^{CL(I)}_{i,J}\right)^2 \frac{s^2_{I-i+k}}{\left(\widehat{f}^{CL(I)}_{I-i+k}\right)^2} \left[\frac{1}{\widehat{C}^{CL(I)}_{i,I-i+k}} + \prod_{m=1}^{k} \left(1-\alpha^{(I)}_{I-i+m}\right) \frac{1}{\sum_{\ell=1}^{i-k-1} C_{\ell,I-i+k}} \right] \qquad (1.4) \\
&\quad + \left(\widehat{C}^{CL(I)}_{i,J}\right)^2 \sum_{j=I-i+k+1}^{J-1} \frac{s^2_j}{\left(\widehat{f}^{CL(I)}_j\right)^2} \left[\alpha^{(I)}_{j-k} \prod_{m=0}^{k-1} \left(1-\alpha^{(I)}_{j-m}\right) \frac{1}{\sum_{\ell=1}^{I-j-1} C_{\ell,j}} \right].
\end{aligned}
$$

$$
\begin{aligned}
\varrho^{(I)}_{I+k+1} &= \widehat{\mathbb{E}}\left[\mathrm{msep}^{\mathrm{MW}}_{\sum_{i=I-J+k+1}^{I} \mathrm{CDR}_{i,I+k+1}|\mathcal{D}_{I+k}}(0) \,\middle|\, \mathcal{D}_I \right] = \sum_{i=I-J+k+1}^{I} \varrho^{(I)}_{i,I+k+1} \qquad (2.4) \\
&\quad + 2 \sum_{I-J+k+1 \le i < n \le I} \widehat{C}^{CL(I)}_{i,J} \widehat{C}^{CL(I)}_{n,J} \frac{s^2_{I-i+k}}{\left(\widehat{f}^{CL(I)}_{I-i+k}\right)^2} \prod_{m=1}^{k} \left(1-\alpha^{(I)}_{I-i+m}\right) \frac{1}{\sum_{\ell=1}^{i-k-1} C_{\ell,I-i+k}} \\
&\quad + 2 \sum_{I-J+k+1 \le i < n \le I} \widehat{C}^{CL(I)}_{i,J} \widehat{C}^{CL(I)}_{n,J} \sum_{j=I-i+k+1}^{J-1} \frac{s^2_j}{\left(\widehat{f}^{CL(I)}_j\right)^2} \left[\alpha^{(I)}_{j-k} \prod_{m=0}^{k-1} \left(1-\alpha^{(I)}_{j-m}\right) \frac{1}{\sum_{\ell=1}^{I-j-1} C_{\ell,j}} \right].
\end{aligned}
$$

Our version (algebraically identical) $\ell = k+1$

$$
\mathrm{MSEP}_{\hat{C}^{[k]}-\hat{C}^{[k+1]},0} \approx \hat{C} \sum_{j=1}^{J} \hat{\rho}_j \left(\frac{1}{1-\hat{q}_{j-k}} - \frac{1}{1-\hat{q}_{j-k-1}} \right)
$$

The Splitting Property

For any m such that $k \leq m \leq \ell$, we obviously have

$$\hat{C} \sum_{j=1}^{J} \hat{\pi}_j \left(\hat{q}_{j-k} - \hat{q}_{j-\ell}\right) = \hat{C} \sum_{j=1}^{J} \hat{\pi}_j \left(\left(\hat{q}_{j-k} - \hat{q}_{j-m}\right) + \left(\hat{q}_{j-m} - \hat{q}_{j-\ell}\right)\right)$$

and

$$\begin{aligned} & \hat{C} \sum_{j=1}^{J} \hat{\rho}_j \left(\frac{1}{1-\hat{q}_{j-k}} - \frac{1}{1-\hat{q}_{j-\ell}}\right) \\ = \quad & \hat{C} \sum_{j=1}^{J} \hat{\rho}_j \left(\left(\frac{1}{1-\hat{q}_{j-k}} - \frac{1}{1-\hat{q}_{j-m}}\right) + \left(\frac{1}{1-\hat{q}_{j-m}} - \frac{1}{1-\hat{q}_{j-\ell}}\right)\right) \end{aligned}$$

hence the cash flow "splits" over sub-periods (no surprise), and so does our MSEP estimator (not trivial!).

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Interpreting the MSEP formula

$$\hat{C} \quad \sum_j \quad \hat{\rho}_j \quad \left(\frac{1}{1-\hat{q}_{j-k}} - \frac{1}{1-\hat{q}_{j-\ell}}\right)$$

Volume Risk Flow Pattern Triangle Geometry

- Risk flow pattern: only depends on CL model parameters; same dimension as $\hat{C}$, e.g. CHF; characteristic of the line of business;
- Influence factors $\hat{q}_j$ do depend on data, but may often be approximated by "geometry". E. g.,

$$\hat{q}_j \approx \frac{j}{J+1}$$

  may be a reasonable average value for roughly constant business volume.
- "Geometric approximation" probably not worse than FINMA "reserve cash flow patterns"

Application to Regulatory Solvency Models

- Current standard regulatory reserve risk models use

$$\text{Reserve Risk} = \text{Reserve} \cdot \alpha, \quad (\text{e.g. } \alpha = 8\%),$$

  - where α is company-individual (hence, non-standard), or
  - the risk does not diversify with volume.

- Our MSEP formula opens up the possibility to use

$$\text{Reserve Risk} = \sqrt{\text{Ultimate} \cdot \beta}, \quad (\text{e.g. } \beta = 250\,000 \text{ CHF}),$$

  which does diversify with volume, and where
  - the result is "fully Merz/Wüthrich compatible";
  - $\beta = \sum_j \hat{\rho}_j ((1 - \hat{q}_j)^{-1} - (1 - \hat{q}_{j-1})^{-1})$ is justifiably "entity-independent":
  - the risk flow pattern $\hat{\rho}_j$ could be prescribed per line of business and
  - the influence factors $\hat{q}_j$ could be estimated "geometrically", possibly taking into account average growth of the business

Application to Run-Off Capital Charge

4370	6293	10292	12460	13660	14307
2701	5291	7162	8945	9338	9780
4483	6729	10074	11142	11971	12538
3254	5804	8351	9874	10608	11111
8010	12118	18028	21315	22901	23986
5582	8864	13187	15592	16752	17546

$k =$	0	1	2	3	4
$m_k =$	3678	2320	1415	724	294
$R_k =$	28430	16444	7532	3039	793
$\frac{m_k}{R_k} =$	12.9%	14.1%	18.8%	23.8%	37.1%

$\sqrt{\text{Total MSEP}} = 4639$ (Today to Ultimate) $= \sqrt{\sum_k m_k^2}$.

$m_k := \sqrt{\text{MSEP}}$ of loss dev. between k and $k+1$ periods from today

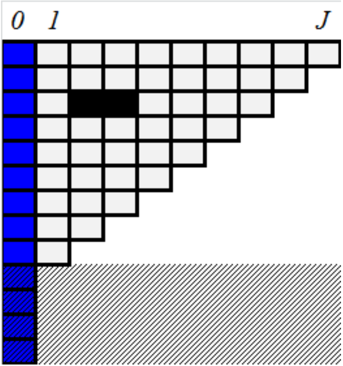
$R_k :=$ reserves at k periods from today

In solvency risk model, might have used 12.9% througout, underestimating the run-off capital charge!

Application to IFRS 17

Not much complexity is added to our MSEP formula by

- allowing "ragged" triangle data; e.g., taking premium (or other volume measure) as first column (blue area) ⟶ integrated view of reserve and premium risk (see also Diers et al. (2016));
- measuring the prediction error only for a subportfolio (shaded area) — splitting off, for example, the premium risk (or the risk adjustment for the remaining coverage under IFRS 17);



- dealing with unreliable, "deleted" data (black entries).

See Röhr (2016) for details and (slightly) generalized formulas.

Application: Aggregate Statistics

From cash flow pattern, get aggregate statistics

- duration
- discount factors

On the risk side, a statistic of interest may be the "total risk flow" $\sum_j \hat{\rho}_j$.
NB: it only captures risk after the end of the first development step, i. e. the column $j = 0$.

If the first development step is the first year loss development, then typical values for the total (reserve) risk flow are:

- order of CHF 10^4: light short tail business
- order of CHF 10^5: medium to long tail business
- order of CHF 10^6: medium or long tail business with large risks

If the first development step is the premium (see previous slide), "premium risk" is included in the risk flow pattern, and these values become considerably larger.

Conclusion

Risk flow patterns...

- ... arise naturally in Mack's stochastic chain ladder framework
- ... help to determine "what part of our insurance risk materializes between k and ℓ years from now?"
  - cost of capital, SST, Solvency II, IFRS 17
- ... may be considered as characteristics of lines of business
  - benchmarking, regulatory use
- ... have nice properties:
  - volume-independent, transform naturally upon change in time granularity

... just like cash flow patterns!!!

References

- Bühlmann, H., De Felice, M., Gisler, A., Moriconi, F., Wüthrich, M. V. (2009) Recursive credibility formula for chain ladder factors and the claims development result. *ASTIN Bulletin* **39(1)**, 275–306.
- Diers, D., Linde, M., Hahn, L. (2016) Quantification of multi-year non-life insurance risk in chain ladder reserving models. *Insurance: Mathematics and Economics* **67**, 187–199.
- Gisler, A. (2016), The Chain Ladder Reserve Uncertainties Revisited, *Paper presented at the ASTIN Colloquium 2016 in Lisbon*.
- Mack, T. (1993) Distribution-free calculation of the standard error of chain ladder reserve estimates. *ASTIN Bulletin* **23(2)**, 213–225.
- Merz, M., Wüthrich, M. V. (2008). Modelling the claims development result for solvency purposes *CAS E-Forum Fall 2008*, 542–568.
- Merz, M., Wüthrich, M. V. (2014) Claims Run-Off Uncertainty: The Full Picture (July 3, 2015). Swiss Finance Institute Research Paper No. 14-69. Available at SSRN: https://ssrn.com/abstract=2524352.
- Röhr, A. (2016) Chain ladder and error propagation, *ASTIN Bulletin*, **46(2)**, 293–330, https://doi.org/10.1017/asb.2016.9.
- Röhr, A. (2016a) Chain ladder prediction error formulae and their interpretation, https://www.researchgate.net/publication/328840909_Chain_Ladder_Prediction_Error_Formulae_and_Their_Interpretation